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# The Rational Defect of a Plane Domain

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Let $K$ be a compact subset of the plane, $\mathbb{C}$, such that $\mathbb{C}\backslash K$ has only a finite number of components, $O_0$, $O_1, \ldots, O_n$; $O_0$ being the unbounded one. Choose $z_i \in O_i$, for $i = 1, 2, \ldots, n$. A theorem of Walsh [4] states that if $u$ is a continuous real valued function on $K$ which is harmonic on the interior of $K$ and if $\epsilon > 0$ is given, then there is a rational function $f$, which is holomorphic on $K$, and real numbers $\alpha_1, \ldots, \alpha_n$, such that

$$\left| u(z) - \operatorname{Re} f(z) - \sum_{i=1}^{n} \alpha_i \log | z - z_i | \right| < \epsilon,$$

for all $z \in K$. Walsh points out that sometimes the logarithmic terms can be dispensed with (as is the case when $K = \{z \in \mathbb{C}: |z| = 1\}$), while in other cases they cannot (as is the case when $K = \{z \in \mathbb{C}: \frac{1}{2} < |z| < 1\}$). The precise determination of which logarithmic terms are necessary was first given in [1]; another proof was given by Glicksberg [2]. The result can be stated as follows: let $L_0$, $L_1, \ldots, L_k$ be the components of the *closure* of $\mathbb{C}\backslash K$, $L_0$ being the unbounded one, choose $z_i$ from the interior of $L_i$ for $i = 1, \ldots, k$. Then if $u$ is a continuous real valued function on $K$ that is harmonic on the interior of $K$ and if $\epsilon > 0$ is given, there is a rational function $f$ that is holomorphic on $K$, and real numbers $\alpha_1, \ldots, \alpha_k$ such that

$$\left| u(z) - \operatorname{Re} f(z) - \sum_{i=1}^{k} \alpha_i \log | z - z_i | \right| < \epsilon,$$

for all $z \in K$.



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Both of the proofs of this result cited above are quite long and involved. They use ideas and techniques from the theory of uniform algebras. The purpose of this note is to give a proof of this theorem which uses only classical results on conformal mappings.

The fact that the number $k$ described above is the least possible was essentially observed by Walsh [4]. In fact, for each $i$, $1 \leq i \leq k$ there is a system $\Gamma_i$ of simple closed curves in the interior of $K$, such that the winding number of $\Gamma_i$ about each point of $L_i$ is 1 and around each point of $L_j$, for $j \neq i$, is 0. Now if $f$ is a rational function that is kolomorphic on $K$ then the change in the harmonic conjugate of $\operatorname{Re} f(z) + \sum_{j \neq i} \alpha_j \log | z - z_j |$ around $\Gamma_i$ is 0, of $\log | z - z_i |$ around $\Gamma_i$ is $2\pi$. It follows that $\log | z - z_i |$ cannot be uniformly approximated on $K$ by functions of the form $\operatorname{Re} f(z) + \sum_{j \neq i} \alpha_j \log | z - z_j |$.

Before proceeding to the proof we need a few more definitions. A bounded open connected set $D \subseteq \mathbb{C}$ is called a ring domain if the boundary of $D$ consists of two disjoint continua. For such a domain $D$ there is a unique real number $r$, $0 < r < 1$, such that $D$ is conformally equivalent to $\Delta_r = \{z \in \mathbb{C}: r < | z | < 1\}$. The modulus of $D$ is defined to be $m(D) = \log(1/r)$. If $D_1$, $D_2$ are ring domains and $D_1 \subseteq D_2$ and $D_1$ separates the bounding continua of $D_2$, then $m(D_1) \leq m(D_2)$. See Tsuji [3, pp. 96–100] for a discussion of ring domains. For any compact set $L \subseteq \mathbb{C}$, let $A(L)$ be the set of functions continuous on $L$ and holomorphic on the interior of $L$. The following lemma is, no doubt, well known.

LEMMA. *Let $D$ be a bounded open set whose boundary is the disjoint union of two simple closed rectifiable curves. Suppose that 0 lies in the bounded component of $\mathbb{C} \setminus \bar{D}$, then*

$$\inf_{g \in A(\bar{D})} \sup_{z \in D} | \log | z | + \operatorname{Re} g(z)| \leq \tfrac{1}{2} m(D).$$

(Actually equality holds but we don't need this fact.)

*Proof.* Let $\Delta = \{z: r < | z | < 1\}$, where $m(D) = \log(1/r)$, and let $\varphi$ be a conformal mapping of $\Delta$ onto $D$. Then $\varphi$ extends to be a homeomorphism of $\bar{\Delta}$ onto $\bar{D}$. It follows that

$$\inf_{g \in A(\bar{D})} \sup_{z \in D} | \log | z | + \operatorname{Re} g(z)| = \inf_{h \in A(\bar{\Delta})} \sup_{\zeta \in \Delta} | \log | \varphi(\zeta)| + \operatorname{Re} h(\zeta)|.$$

Now since $\varphi$ is one to one and 0 is in the bounded component of $\mathbb{C} \setminus \bar{D}$ it follows that

$$\varphi(z) = z^{\epsilon} \exp k(z),$$

where $k$ is holomorphic in $\Delta$ and $\epsilon = \pm 1$. Since $\varphi$ is bounded away from 0 and lies in $A(\bar{\Delta})$ it follows that $k \in A(\bar{\Delta})$. Hence we see that

$$\inf_{h \in A(\bar{\Delta})} \sup_{z \in \Delta} |\log|\varphi(z)| + \operatorname{Re} h(z)|$$

$$= \inf_{h \in A(\bar{\Delta})} \sup_{z \in \Delta} |\log|z| + \operatorname{Re} h(z)|$$

$$\leq \sup_{z \in \Delta} \left| \log|z| - \frac{\log r}{2} \right| = \frac{1}{2} \log \frac{1}{r} = \frac{1}{2} m(D).$$

The result we are after is a consequence of the following

THEOREM. *Let $K \subseteq \mathbb{C}$ be compact and let $U$, $V$ be components of $\mathbb{C} \backslash K$ such that $\bar{U} \cap \bar{V} \neq \varnothing$. If $a \in U$ and $b \in V$ and if $\epsilon > 0$ is given then there is a rational function $f$, holomorphic on $K$, such that*

$$\left| \operatorname{Re} f(z) - \log \left| \frac{z-a}{z-b} \right| \right| < \epsilon, \qquad \textit{for all} \quad z \in K.$$

*Proof.* Using a linear fractional transformation we map $a$ to 0 and $b$ to $\infty$. Then we have the following problem: if $U$, $V$ are components of $\mathbb{C} \backslash K$ such that $0 \in U$ and $V$ is unbounded and $\bar{U} \cap \bar{V} \neq \varnothing$, then if $\epsilon > 0$ is given there is a rational function $f$, holomorphic on $K$, such that

$$|\log|z| - \operatorname{Re} f(z)| < \epsilon, \qquad \text{for all} \quad z \in K.$$

Take $\alpha_n \in U$ and $\beta_n \in V$ such that $|\alpha_n - \beta_n| \to 0$. Let $\gamma_n$ be a simple closed rectifiable curve in $U$ that contains 0 and $\alpha_n$ in its interior, and let $\Gamma_n$ be a simple closed rectifiable curve in $V$ that contains $K$ in its interior and $\beta_n$ in its exterior. Let $D_n$ be the ring domain bounded by $\gamma_n$ and $\Gamma_n$. We choose $\gamma_n$, $\Gamma_n$ in such a way that $\bar{D}_{n-1} \subseteq D_n$ and $D_{n+1}$ separates the boundary curves of $D_n$. If $R(K)$ denotes the space of rational functions with poles off $K$, then it follows from Runge's Theorem and the lemma that

$$\inf_{h \in R(K)} \sup_{z \in K} |\log|z| + \operatorname{Re} h(z)|$$

$$\leq \inf_{g \in A(\bar{D}_n)} \sup_{z \in D_n} |\log|z| + \operatorname{Re} g(z)| \leq \tfrac{1}{2} m(D_n),$$

for every $n$. So it is sufficient to show that $m(D_n) \to 0$ as $n \to \infty$. To this end, let $m(D_n) = \log(1/r_n)$, and $\Delta_n = \{z\colon r_n < |z| < 1\}$. Since $D_n \supseteq D_{n+1}$ and $D_{n+1}$ separates the bounding curves of $D_n$, $1 > r_{n+1} > r_n$. Suppose $r_n \to r_0 < 1$ as $n \to \infty$, and denote $\{z\colon r_0 < |z| < 1\}$ by $\Delta_0$. Let $\varphi_n$ be a conformal map of $\Delta_n$ onto $D_n$, by passing to a subsequence we may assume that $\{\varphi_n\}$ converges uniformly on compact subsets of $\Delta_0$ to a function $\varphi$ which is bounded away from 0. Fix $r$, $r_0 < r < 1$, and let $\gamma(t) = re^{it}$, $0 \leqslant t \leqslant 2\pi$. For every $n$, $\varphi_n \circ \gamma$ has winding number $+1$ about the origin, hence $\varphi \circ \gamma$ has winding number $+1$ about the origin. In particular $\varphi$ is not constant. If $z \in \Delta_0$ then $\varphi(z) = \lim \varphi_n(z) \in \bigcap_{n=1} D_n = L$. But since $\varphi$ is not constant it is an open map and hence $\varphi(\Delta_0) \subseteq \operatorname{int} L$. In particular $\varphi \circ \gamma$ is a curve in the interior of $L$ that has winding number $+1$ about the origin. However it is clear from the construction of $L$ that every component of the interior of $L$ is simply connected. This contradiction shows that $m(D_n) \to 0$ and this completes the proof of the theorem.

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